Function of a function

If y is a function of x then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

This is known as the **'function of a function'** rule (or sometimes the **chain rule**).
For example, if $y = (3x - 1)^9$ then, by making the substitution $u = (3x - 1)$, $y = u^9$, which is of the 'standard' form.

Hence $\dfrac{dy}{du} = 9u^8$ and $\dfrac{du}{dx} = 3$

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx} = (9u^8)(3) = 27u^8$

Rewriting u as $(3x - 1)$ gives: $\dfrac{dy}{dx} = \mathbf{27(3x - 1)^8}$

EXAMPLE 1 Differentiate $y = 3\cos(5x^2 + 2)$

Let $u = 5x^2 + 2$ then $y = 3\cos u$

Hence $\dfrac{du}{dx} = 10x$ and $\dfrac{dy}{du} = -3\sin u$

Using the function of a function rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (-3\sin u)(10x) = -30x \sin u$$

Rewriting u as $5x^2 + 2$ gives:

$$\frac{dy}{dx} = -30x \sin(5x^2 + 2)$$

EXAMPLE 2 Find the derivative of $y = (4t^3 - 3t)^6$

SOLUTION

Let $u = 4t^3 - 3t$, then $y = u^6$

Hence $\dfrac{du}{dt} = 12t^2 - 3$ and $\dfrac{dy}{dt} = 6u^5$

Using the function of a function rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (6u^5)(12t^2 - 3)$$

Rewriting u as $(4t^3 - 3t)$ gives:

$$\frac{dy}{dt} = 6(4t^3 - 3t)^5(12t^2 - 3)$$
$$= \mathbf{18(4t^2 - 1)(4t^3 - 3t)^5}$$

EXAMPLE 2 Differentiate $y = \sqrt{3x^2 + 4x - 1}$

SOLUTION

$y = \sqrt{3x^2 + 4x - 1} = (3x^2 + 4x - 1)^{1/2}$

Let $u = 3x^2 + 4x - 1$ then $y = u^{1/2}$

Hence $\frac{du}{dx} = 6x + 4$ and $\frac{dy}{du} = \frac{1}{2}u^{-1/2} = \frac{1}{2\sqrt{u}}$

Using the function of a function rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \left(\frac{1}{2\sqrt{u}}\right)(6x + 4) = \frac{3x + 2}{\sqrt{u}}$$

i.e. $\frac{dy}{dx} = \frac{3x + 2}{\sqrt{3x^2 + 4x - 1}}$

EXAMPLE 3 Differentiate

$$y = \frac{2}{(2t^3 - 5)^4}$$

SOLUTION

$y = \frac{2}{(2t^3 - 5)^4} = 2(2t^3 - 5)^{-4}$. Let $u = (2t^3 - 5)$,
then $y = 2u^{-4}$

Hence $\frac{du}{dt} = 6t^2$ and $\frac{dy}{du} = -8u^{-5} = \frac{-8}{u^5}$

Then $\frac{dy}{dt} = \frac{dy}{du} \times \frac{du}{dt} = \left(\frac{-8}{u^5}\right)(6t^2) = \frac{-48t^2}{(2t^3 - 5)^5}$

CLASS WORK Find the differential coefficients with respect to the variables

1. $(2x^3 - 5x)^5$ $[5(6x^2 - 5)(2x^3 - 5x)^4]$
2. $2\sin(3\theta - 2)$ $[6\cos(3\theta - 2)]$

ASSIGNMENT Differentiate the variables

$2\cot(5t^2 + 3)$ $[-20t\,\mathrm{cosec}^2(5t^2 + 3)]$

$6\tan(3y + 1)$ $[18\sec^2(3y + 1)]$